MORE ILLON Philippe SF2940 Homework 2

Problem 1

(a) Let $X \in Exp(1)$ and $Y \in Exp(1)$ be independent random variables

Do $\frac{X+Y}{2}$ and $\max(X,Y)$ have the same distribution ?

Fix $u \geq 0$.

$$\mathbb{P}\left(\frac{X+Y}{2} \leq u\right) = \mathbb{P}(X+Y \leq 2u) = \int_0^{2u} f_{X+Y}(v)\,dv = \int_0^{2u}\int_0^{+\infty} f_X(v-y) f_Y(y)\,dy\,dv$$

$$= \int_0^{2u}\int_0^{+\infty} e^{-(v-y)} \mathbb{1}_{\{v-y\geq 0\}} e^{-y} \mathbb{1}_{\{y \geq 0\}}\,dy\,dv = \int_0^{2u}\int_0^{v} e^{-v}\,dy\,dv = \int_0^{2u} e^{-v}\cdot v\,dv = -e^{-v}\cdot v\Big|_0^{2u} + \int_0^{2u} e^{-v}\,dv$$

$$= -e^{-2u}\cdot 2u + (-e^{-v})\Big|_0^{2u} = -e^{-2u}\cdot 2u + 1 - e^{-2u} = 1 - e^{-2u}(1+2u)$$

Now $\mathbb{P}(\max(X,Y) \leq u) = \mathbb{P}(\{X \leq u\}\cap\{Y \leq u\}) \overset{ind}{=} \mathbb{P}(X \leq u)\mathbb{P}(Y \leq u) = \left(\int_0^u e^{-x}\,dx\right)^2 = \left(-e^{-x}\Big|_0^u\right)^2 = (1-e^{-u})^2$

$\Rightarrow \mathbb{P}\left(\frac{X+Y}{2} \leq u\right) \neq \mathbb{P}(\max(X,Y) \leq u)$, so they are not identically distributed.

(b) Let $X_1, X_2, \dots, X_n, \dots$ be iid $N(\mu, \sigma^2)$ - random variables with $\mu \neq 0$.

To determine the asymptotic distribution of $\left(\left(\frac{1}{n}\sum_{i=1}^n X_i\right)^2 - \mu^2\right)\sqrt{n}$.

We want to use the law of large numbers and the central limit theorem :

$$\sqrt{n}\left(\left(\frac{1}{n}\sum_{i=1}^n X_i\right)^2 - \mu^2\right) = \underbrace{\sqrt{n}\left(\frac{1}{n}\sum_{i=1}^n X_i - \mu\right)}_{\overset{\mathbb{P}}{\to} N(0,\sigma^2) \text{ as } n\to+\infty}\underbrace{\left(\frac{1}{n}\sum_{i=1}^n X_i + \mu\right)}_{\overset{\mathbb{P}}{\to} 2\mu \text{ as } n\to\infty}$$

(by the law of large numbers)

(by the central limit theorem)

By Slutsky, $\sqrt{n}\left(\left(\frac{1}{n}\sum_{i=1}^n X_i\right)^2 - \mu^2\right) \overset{\mathbb{P}}{\to} 2\mu \cdot N(0,\sigma^2) = N(0, 4\mu^2\sigma^2)$ as $n \to +\infty$

$$\Rightarrow \sqrt{n}\left(\left(\frac{1}{n}\sum_{i=1}^n X_i\right)^2 - \mu^2\right) \overset{d}{\to} N(0, 4\mu^2\sigma^2) \text{ as } n \to +\infty.$$

Problem 2 : Let $(X_n)_{n\geq 1}$; $\mathbb{P}(X_n = 1) = \mathbb{P}(X_n = -1) = \frac{1}{2}$.

Let $N \in Po(\lambda)$ and independent of $(X_n)_{n\geq1}$. Set $S_N = X_1^3 + X_2^3 + \dots + X_N^3$, $S_0 = 0$.

To show : $\frac{S_N}{\sqrt{\lambda}} \overset{d}{\to} N(0,1)$ as $\lambda \to +\infty$.

By the course, $\left\{\frac{S_N}{\sqrt{\lambda}} \overset{d}{\to} N(0,1) \text{ as } \lambda\to+\infty\right\} \Leftrightarrow \left\{\mathbb{E}\left[e^{it\frac{S_N}{\sqrt{\lambda}}}\right] \to e^{-t^2/2} \text{ as } \lambda\to\infty \;\; \forall t \in \mathbb{R}\right\}$

$$\mathbb{E}\left[e^{it\frac{S_N}{\sqrt{\lambda}}}\right] = \mathbb{E}\left[\mathbb{E}\left[e^{it\frac{S_N}{\sqrt{\lambda}}} \Big| N\right]\right] \overset{N \text{ ind. of } (X_j)}{=} \mathbb{P}(N=0) + \sum_{k\geq1}\mathbb{E}\left[e^{it\frac{S_k}{\sqrt{\lambda}}}\right]\mathbb{P}(N=k) = \mathbb{P}(N=0) + \sum_{k\geq1}\mathbb{E}\left[\prod_{j=1}^k e^{itX_j^3/\sqrt{\lambda}}\right]\mathbb{P}(N=k)$$

$$\overset{(X_j) \text{ iid}}{=} \mathbb{P}(N=0) + \sum_{k\geq1}\left(\mathbb{E}\left[e^{itX_1^3/\sqrt{\lambda}}\right]\right)^k\mathbb{P}(N=k) \underset{X_1^3 = X_1 \text{ since } 1^3=1 \text{ and } (-1)^3=-1.}{=} \mathbb{P}(N=0) + \sum_{k\geq1}\left(\mathbb{E}\left[e^{itX_1/\sqrt{\lambda}}\right]\right)^k\mathbb{P}(N=k) \quad (*)$$

Now, $\mathbb{E}[e^{itX_1/\sqrt{\lambda}}] = e^{it/\sqrt{\lambda}}\cdot\frac{1}{2} + e^{-it/\sqrt{\lambda}}\cdot\frac{1}{2} = \cos(t/\sqrt{\lambda})$

$\Rightarrow (*) = e^{-\lambda} + \sum_{k\geq 1} (\cos(t/\sqrt{\lambda}))^k e^{-\lambda}\cdot\frac{\lambda^k}{k!} = e^{-\lambda}\left(1+\sum_{k\geq 1}\frac{(\cos(t/\sqrt{\lambda})\cdot\lambda)^k}{k!}\right) = e^{-\lambda}\cdot e^{\cos(t/\sqrt{\lambda})\cdot\lambda} = e^{\lambda(\cos(t/\sqrt{\lambda})-1)}$

Now we want to determine $\lim_{\lambda\to\infty} \lambda(\cos(t/\sqrt{\lambda})-1)$. $\lambda(\cos(t/\sqrt{\lambda})-1) = \lambda\left(\sum_{k\geq 0}\frac{(t/\sqrt{\lambda})^{2k}}{(2k)!}(-1)^k - 1\right)$

$= \lambda\sum_{k\geq 1}\left(\frac{t}{\sqrt{\lambda}}\right)^{2k}\cdot\frac{(-1)^k}{(2k)!} = -\frac{t^2}{2} + \underbrace{\sum_{k\geq 2}\lambda\left(\frac{t}{\sqrt{\lambda}}\right)^{2k}\cdot\frac{(-1)^k}{(2k)!}}_{\to 0 \text{ as } \lambda\to\infty}$ $\Rightarrow \lim_{\lambda\to+\infty}\mathbb{E}[e^{itS_\lambda/\sqrt{\lambda}}] = e^{-t^2/2}$, the characteristic function of a $\mathcal{N}(0,1)$-distributed random variable.

Problem 3 Let $X=\binom{X_1}{X_2}\in\mathcal{N}(\mu,\Lambda)$, where $\mu=\binom{1}{1}$ and $\Lambda=\begin{pmatrix}3&1\\1&2\end{pmatrix}$

(1) Compute $\mathbb{P}(X_1\geq 2 \mid X_2+3X_1=3)$

Let's first compute $f_{X_1|3X_1+X_2=3}(x_1) := \dfrac{f_{(X_1,3X_1+X_2)}(x_1,3)}{f_{3X_1+X_2}(3)}$

Lemma (course): $X\in\mathcal{N}(\mu,\Lambda) \Leftrightarrow \alpha_1X_1+\alpha_2X_2$ is normally-distributed, $\forall\alpha_1,\alpha_2\in\mathbb{R}$

$\Rightarrow 3X_1+X_2$ is normally distributed and $\binom{X_1}{3X_1+X_2}$ is normally distributed.

Hence we just have to compute the covariance matrix and the mean vector of $\binom{X_1}{3X_1+X_2}$ to determine the law.

Reminder: Let A be a 2×2-matrix. $\mathbb{E}[A\binom{X_1}{X_2}] = A\,\mathbb{E}[\binom{X_1}{X_2}]$ by linearity

$Cov(A\binom{X_1}{X_2}) = \mathbb{E}\left[\left(A\binom{X_1}{X_2}-\mathbb{E}[A\binom{X_1}{X_2}]\right)\left(A\binom{X_1}{X_2}-\mathbb{E}[A\binom{X_1}{X_2}]\right)^T\right]$

$= \mathbb{E}\left[A\left(\binom{X_1}{X_2}-\mathbb{E}[\binom{X_1}{X_2}]\right)\left(\binom{X_1}{X_2}-\mathbb{E}[\binom{X_1}{X_2}]\right)^T A^T\right] = A\, Cov\binom{X_1}{X_2}A^T$

Now $\binom{X_1}{3X_1+X_2} = \begin{pmatrix}1&0\\3&1\end{pmatrix}\binom{X_1}{X_2} \Rightarrow \mathbb{E}[\binom{X_1}{3X_1+X_2}] = \begin{pmatrix}1&0\\3&1\end{pmatrix}\mathbb{E}[\binom{X_1}{X_2}] = \begin{pmatrix}1&0\\3&1\end{pmatrix}\binom{1}{1} = \binom{1}{4}$

and $Cov(\binom{X_1}{3X_1+X_2}) = Cov(\begin{pmatrix}1&0\\3&1\end{pmatrix}\binom{X_1}{X_2}) = \begin{pmatrix}1&0\\3&1\end{pmatrix}Cov(\binom{X_1}{X_2})\begin{pmatrix}1&0\\3&1\end{pmatrix}^T = \begin{pmatrix}1&0\\3&1\end{pmatrix}Cov(\binom{X_1}{X_2})\begin{pmatrix}1&3\\0&1\end{pmatrix}$

$=\begin{pmatrix}1&0\\3&1\end{pmatrix}\begin{pmatrix}3&1\\1&2\end{pmatrix}\begin{pmatrix}1&3\\0&1\end{pmatrix} = \begin{pmatrix}1&0\\3&1\end{pmatrix}\begin{pmatrix}3&10\\1&5\end{pmatrix} = \begin{pmatrix}3&10\\10&35\end{pmatrix}$

$\Rightarrow \binom{X_1}{3X_1+X_2}\in\mathcal{N}\left(\binom{1}{4}, \begin{pmatrix}3&10\\10&35\end{pmatrix}\right)$ and $3X_1+X_2\in\mathcal{N}(4,35)$

$\Rightarrow f_{(X_1,3X_1+X_2)}(x_1,3) = \frac{1}{2\pi\sqrt{5}} e^{-\frac{1}{2}\left(\binom{x_1}{3}-\binom{1}{4}\right)^T\frac{1}{5}\begin{pmatrix}35&-10\\-10&3\end{pmatrix}\left(\binom{x_1}{3}-\binom{1}{4}\right)}$

$\det\begin{pmatrix}3&10\\10&35\end{pmatrix} = 5$

$\begin{pmatrix}3&10\\10&35\end{pmatrix}^{-1} = \frac{1}{5}\begin{pmatrix}35&-10\\-10&3\end{pmatrix}$

and $\left(\binom{x_1}{3}-\binom{1}{4}\right)^T\frac{1}{5}\begin{pmatrix}35&-10\\-10&3\end{pmatrix}\left(\binom{x_1}{3}-\binom{1}{4}\right) =$

$(x_1-1 \quad -1)\,\frac{1}{5}\begin{pmatrix}35&-10\\-10&3\end{pmatrix}\binom{x_1-1}{-1} = 7(x_1-1)^2+4(x_1-1)+\frac{3}{5}$

$= 7(x_1^2-2x_1+1)+4x_1-4+\frac{3}{5} = 7x_1^2-10x_1+\frac{18}{5}$

$$\Rightarrow f_{(X_1, 3X_1+X_2)}(x_1, 3) = \frac{1}{2\pi\sqrt{5}} e^{-\frac{1}{2}\left(7x_1^2 - 10x_1 + \frac{18}{5}\right)}$$

$$f_{3X_1+X_2}(3) = \frac{1}{\sqrt{2\pi}\sqrt{35}} e^{-\frac{1}{2}\left(\frac{(3-4)^2}{35}\right)} = \frac{1}{\sqrt{2\pi\cdot 35}} e^{-\frac{1}{2}\left(\frac{1}{35}\right)}$$

Now $7x_1^2 - 10x_1 + \frac{18}{5} - \frac{1}{35} = 7x_1^2 - 10x_1 + \frac{125}{35} = 7x_1^2 - 10x_1 + \frac{25}{7} = 7\left(x_1 - \frac{5}{7}\right)^2$

$$\Rightarrow f_{X_1|3X_1+X_2=3}(x_1) = \frac{1}{\sqrt{2\pi}\frac{1}{\sqrt{7}}} e^{-\frac{1}{2}\left(\frac{(x_1-5/7)^2}{1/7}\right)} \Rightarrow X_1|3X_1+X_2=3 \in \mathcal{N}\left(\frac{5}{7}, \frac{1}{7}\right)$$

Now we want to express $\mathbb{P}(X_1 \geq 2 / 3X_1 + X_2 = 3)$ as a function of $\Phi(t) := \mathbb{P}(X \leq t)$, where $X \in \mathcal{N}(0,1)$.

Let $Y \in \mathcal{N}\left(\frac{5}{7}, \frac{1}{7}\right)$. $\mathbb{P}(X_1 \geq 2 / 3X_1 + X_2 = 3) = \mathbb{P}(Y \geq 2) = \mathbb{P}\left(\frac{Y - \frac{5}{7}}{\sqrt{\frac{1}{7}}} \geq \frac{2 - 5/7}{\sqrt{1/7}}\right) \underset{X \in \mathcal{N}(0,1)}{=} \mathbb{P}(X \geq 9/\sqrt{7}) = \underline{1 - \Phi\left(\frac{9}{\sqrt{7}}\right)}$

(2) Find $\mathbb{E}[X_1^2 X_2^8 / X_2 = 2]$

As before, we first need to compute $\frac{f_{(X_1,X_2)}(x_1, 2)}{f_{X_2}(2)}$. $(X_1, X_2) \in \mathcal{N}\left(\binom{1}{1}, \begin{pmatrix} 3 & 1 \\ 1 & 2 \end{pmatrix}\right)$ and $X_2 \in \mathcal{N}(1, 2)$

$$f_{X_1|X_2=2}(x_1) \overset{def}{=} \frac{\frac{1}{2\pi\sqrt{5}} e^{-1/2\left(\left(\binom{x_1}{2} - \binom{1}{1}\right)^T \frac{1}{5}\begin{pmatrix} 2 & -1 \\ -1 & 3 \end{pmatrix}\left(\binom{x_1}{2} - \binom{1}{1}\right)\right)}}{\frac{1}{\sqrt{2\pi}\sqrt{2}} e^{-1/2\left(\frac{(2-1)^2}{2}\right)}} = \frac{\sqrt{2}}{\sqrt{2\pi}\sqrt{5}} e^{-1/2\left(\frac{1}{5}(2x_1^2 - 6x_1 + 7) - \frac{1}{2}\right)}$$

$$\left(\binom{x_1}{2} - \binom{1}{1}\right)^T \frac{1}{5}\begin{pmatrix} 2 & -1 \\ -1 & 3 \end{pmatrix}\left(\binom{x_1}{2} - \binom{1}{1}\right) =$$

$$\frac{1}{5}\left(2(x_1 - 1)^2 - 2(2-1)(x_1 - 1) + 3(2-1)^2\right) =$$

$$\frac{1}{5}\left(2x_1^2 - 4x_1 + 2 - 2x_1 + 2 + 3\right) = \frac{1}{5}(2x_1^2 - 6x_1 + 7)$$

$$= \frac{\sqrt{2}}{\sqrt{2\pi}\sqrt{5}} e^{-1/2\left(\frac{1}{5}\left(2x_1^2 - 6x_1 + 7 - \frac{5}{2}\right)\right)} = \frac{\sqrt{2}}{\sqrt{2\pi}\sqrt{5}} e^{-1/2\left(\frac{2}{5}\left(x_1^2 - 3x_1 + \frac{9}{4}\right)\right)} = \frac{\sqrt{2}}{\sqrt{2\pi}\sqrt{5}} e^{-1/2\left(\frac{2}{5}\left(x_1 - \frac{3}{2}\right)^2\right)}$$

Now $\mathbb{E}[X_1^2 X_2^8 / X_2 = 2] = \mathbb{E}[X_1^2 \cdot 2^8 / X_2 = 2] = 2^8\, \mathbb{E}[X_1^2 / X_2 = 2] = 2^8\big(\underbrace{Var(X_1 / X_2 = 2) + \mathbb{E}[X_1 / X_2 = 2]^2}\big)$

We know these two quantities by the computation of $f_{X_1|X_2=2}(x_1)$

$= 2^8\left(\frac{5}{2} + \frac{9}{4}\right) = \underline{2^6 \cdot 19}$

Remark : $U / V = k \sim \mathcal{N}\left(\mu_U - \frac{c_{UV}}{c_V}(\mu_V - k),\ c_U - \frac{c_{UV}^2}{c_V}\right)$

Problem 3 (Exam 2017)

Let $X_1, X_2, \dots$ be $U(0,1)$-distributed iid random variables and $N \in Po(\lambda)$ be independent of $X_1, X_2, \dots$
Set $Y_N = \max\{X_1, X_2, \dots, X_N\}$ and $Y_0 = 0$.

(a.) Fix $u \in (0,1)$. $\mathbb{P}(Y_N \leq u) = \sum_{k\geq 0} \mathbb{P}(Y_N \leq u \mid N=k)\,\mathbb{P}(N=k) = \sum_{k\geq 0} \mathbb{P}(Y_k \leq u \mid N=k)\,\mathbb{P}(N=k)$

$\overset{N \text{ ind.}}{=} \sum_{k\geq 0} \mathbb{P}(Y_k \leq u)\,\mathbb{P}(N=k) = \underbrace{\mathbb{P}(0 \leq u)}_{=1 \text{ since } u\in(0,1)}\mathbb{P}(N=0) + \sum_{k\geq 1} \mathbb{P}(Y_k \leq u)\,\mathbb{P}(N=k) = \mathbb{P}(N=0) + \sum_{k\geq 1} \mathbb{P}(\max\{X_1, \dots, X_k\} \leq u)\,\mathbb{P}(N=k)$

$= \mathbb{P}(N=0) + \sum_{k\geq 1} \mathbb{P}(X_i \leq u\ \forall i \leq k)\,\mathbb{P}(N=k) \overset{X_i \text{ iid}}{=} \mathbb{P}(N=0) + \sum_{k\geq 1} \mathbb{P}(X_1 \leq u)^k\,\mathbb{P}(N=k) = e^{-\lambda} + \sum_{k\geq 1} u^k \frac{\lambda^k e^{-\lambda}}{k!}$

$= e^{-\lambda}\left(1 + \sum_{k\geq 1} \frac{(u\lambda)^k}{k!}\right) = e^{-\lambda}\cdot e^{u\lambda} = e^{-\lambda(1-u)}$

Now for the characteristic function: we want to find $f_{Y_N}(u)$ first

$f_{Y_N}(u) = \frac{d}{du}\mathbb{P}(Y_N \leq u) = \frac{d}{du} e^{-\lambda(1-u)} = e^{-\lambda(1-u)}\cdot \lambda$

Now notice that $\mathbb{P}(Y_N = 0) = \mathbb{P}(N=0) = e^{-\lambda} > 0$.

Hence $\mathbb{E}[e^{itY_N}] = \mathbb{E}[e^{itY_N}\mathbb{1}_{\{N=0\}}] + \mathbb{E}[e^{itY_N}\mathbb{1}_{\{N>0\}}] = \mathbb{P}(N=0) + \int_0^1 e^{itu}\lambda e^{-\lambda(1-u)}\,du =$

$e^{-\lambda} + \int_0^1 \lambda e^{itu-\lambda(1-u)}\,du = e^{-\lambda}\left(1 + \int_0^1 \lambda e^{itu+u}\,du\right) = e^{-\lambda}\left(1 + \int_0^1 \lambda e^{u(it+1)}\,du\right) = e^{-\lambda}\left(1 + \frac{\lambda}{it+1} e^{u(it+1)}\Big|_0^1\right)$

$= e^{-\lambda}\left(1 + \frac{\lambda}{it+1}e^{it+1} - \frac{\lambda}{it+1}\right)$

(b.) $\mathbb{E}[Y_N] = \mathbb{E}[Y_N\mathbb{1}_{\{N=0\}}] + \mathbb{E}[Y_N\mathbb{1}_{\{N>0\}}] = \mathbb{P}(N=0) + \int_0^1 \mathbb{P}(Y_N > t)\,dt = \mathbb{P}(N=0) + \int_0^1 (1 - \mathbb{P}(Y_N \leq t))\,dt$

$= e^{-\lambda} + \int_0^1 (1 - e^{-\lambda(1-t)})\,dt \xrightarrow{\lambda\to+\infty} 1 + 0 = 1.$

(c.) Show that $\lambda(1-Y_N)$ converges in distribution as $\lambda \to +\infty$ and determine the limiting distribution.

$\mathbb{P}(\lambda(1-Y_N) \leq u) = \mathbb{P}(\lambda - \lambda Y_N \leq u) = \mathbb{P}(-\lambda Y_N \leq u - \lambda) = \mathbb{P}\left(Y_N \geq \frac{\lambda-u}{\lambda}\right) = 1 - \mathbb{P}\left(Y_N \leq \frac{\lambda-u}{\lambda}\right)$

$= 1 - e^{-\lambda\left(1-\frac{\lambda-u}{\lambda}\right)} = 1 - e^{-\lambda\left(\frac{u}{\lambda}\right)} = 1 - e^{-u} = \mathbb{P}(X \leq u)$, where $X \in Exp(1)$

Hence it trivially converges to $Exp(1)$ as $\lambda \to +\infty$.

QED

Problem 4

Let $\varphi(t,s,u) = \exp\{2is - s^2 - 2t^2 - 4u^2 - 2st + 2su\}$ be the characteristic function of $\begin{pmatrix} X \\ Y \\ Z \end{pmatrix}$.

Determine the distribution of $\begin{pmatrix} X \\ Y \\ Z \end{pmatrix}$.

Let $\vec{V}$ be a Gaussian vector. Then $\varphi_{\vec{V}}(t,s,u) = \exp\{i\,(t\ s\ u)\cdot\mu - \frac{1}{2}(t\ s\ u)\, C \begin{pmatrix} t \\ s \\ u \end{pmatrix}\}$.

We are going to find μ and C so that $\varphi(t,s,u) = \varphi_{\vec{V}}(t,s,u)$.

$$\varphi_{\vec{V}}(t,s,u) = \exp\{it\mu_1 + is\mu_2 + iu\mu_3 - \frac{1}{2}\left(c_{11}t^2 + c_{22}s^2 + c_{33}u^2 + 2c_{12}ts + 2c_{23}su + 2c_{13}tu\right)\}$$

$\Rightarrow \mu_1 = 0,\ \mu_2 = 2,\ \mu_3 = 0,\ c_{11} = 4,\ c_{22} = -2,\ c_{33} = 8,\ c_{12} = 2,\ c_{13} = 0,\ c_{23} = -2.$

$$\Rightarrow \begin{pmatrix} X \\ Y \\ Z \end{pmatrix} \in N\left(\begin{pmatrix} 0 \\ 2 \\ 0 \end{pmatrix}, \begin{pmatrix} 4 & 2 & 0 \\ 2 & -2 & -2 \\ 0 & -2 & 8 \end{pmatrix}\right)$$

MOREILLON Philippe 078940 (Additional Exercise)

[T.K] 3.8.3.11 Let $X \in Exp(1)$, $Y \in Exp(1)$ be independent. To show: $\frac{X}{X+Y} \in U(0,1)$.

Fix $u \geq 0$.

$$\mathbb{P}\left(\frac{X}{X+Y} \leq u\right) = \mathbb{P}(X \leq u(X+Y)) = \mathbb{P}(X(1-u) \leq uY) = \mathbb{P}\left(X \leq \frac{u}{1-u} Y\right)$$

(Notice that $u < 1$, otherwise $\frac{u}{1-u} \leq 0$ and X is negative! which has $\mathbb{P} = 0$)

$X+Y \geq 0$ since they are $Exp(1)$

$$= \int_0^{+\infty}\int_0^{+\infty} \mathbb{1}_{\{X \leq \frac{u}{1-u} Y\}} f_X(x) f_Y(y)\, dx\, dy = \int_0^{+\infty}\int_0^{\frac{u}{1-u}y} f_X(x) f_Y(y)\, dx\, dy = \int_0^{+\infty}\int_0^{\frac{u}{1-u}y} e^{-x} e^{-y}\, dx\, dy$$

$$= \int_0^{+\infty} e^{-y} \left\{ \int_0^{\frac{u}{1-u}y} e^{-x}\, dx \right\} dy = \int_0^{+\infty} e^{-y} \left\{ -e^{-x} \Big|_0^{\frac{u}{1-u}y} \right\} dy = \int_0^{+\infty} e^{-y}\left(1 - e^{-\frac{u}{1-u}y}\right) dy$$

$$= \int_0^{+\infty} e^{-y}\, dy - \int_0^{+\infty} e^{-y\left(1+\frac{u}{1-u}\right)}\, dy = \int_0^{+\infty} e^{-y}\, dy - \int_0^{+\infty} e^{-y\left(\frac{1}{1-u}\right)}\, dy$$

$$= -e^{-y}\Big|_0^{+\infty} + (1-u)\, e^{-y\left(\frac{1}{1-u}\right)}\Big|_0^{+\infty} = 1 - (1-u) = u \quad \forall u \in (0,1).$$

$$\Rightarrow \frac{X}{X+Y} \in U(0,1)\ !$$

QED